The Gauss map of surfaces into $\mathbb{R}^3$

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My thanks for the invitation to talk in the workshop

Geometric Analysis on Riemannian and
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Basic tools

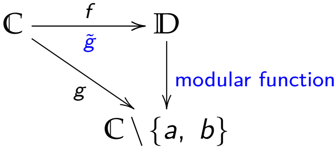
Liouville's theorem:

Every bounded holomorphic function $f: \mathbb{C} \to \mathbb{C}$ is constant. First

Picard theorem:

If $g: \mathbb{C} \to \mathbb{C}$ is holomorphic and $\sharp\Big(\mathbb{C} \setminus f(\mathbb{C})\Big) \geq 2$ then f is constant.

Liouville's theorem $\iff$ first Picard's theorem



Relation with Minimal Surfaces

Theorem (Weierstrass-Enneper-Osserman's Representation)

Let $X : M \hookrightarrow \mathbb{R}^3$ be a complete minimal immersion of finite total curvature. Then there are a compact Riemann surface $\overline{M}$ and a finite set $E = \{w_1, w_2, \ldots, w_n\} \subset \overline{M}$ such that M is conformally to $\overline{M} \setminus E$. The Gauss map g extends to meromorphic function on $\overline{M}$ and there exit a meromorphic 1-form ω on $\overline{M}$ satisfying the following conditions:

1. *z_0 is polo of order n of g, iff z_0 is a zero of order $2n$ of ω*
2. *For all closed path α on M we have $\Re \int_\alpha \phi = 0$*
3. *$\phi = (\frac{1}{2}(1-g^2)\omega, \frac{i}{2}(1+g^2)\omega, g\omega)$ have polo of order greater or equal to 2 for each w_i*

and $X(z) = X(z_0) + \Re \int_{z_0}^{z} \phi$. Reciprocally. All complete minimal immersions with finite total curvature are of this type.

Example n-catenoid Miyaoka-Sato and Yi Fang.

Example (1)

$\overline{M} = \mathbb{C} \setminus \{z \mid |z|^{n+1} = 1\}$ and $g = z^n, \ \omega = fdz, \ f = (z^{n+1} - 1)^{-2}$

Example (2)

Miyaoka - Sato:

There are complete minimal surfaces conformal to $T_g^2 \setminus X$, with genus $g = 0, \ 1, \ 2$, whose Gauss map miss 2 points

Example (3)

Yi Fang:

There are periodic minimal Tori with Gauss map missing 3 points.

Costa-Hoffman-Meeks' surfaces: Embedded.

Example (4)

Consider $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ and

$$\overline{M}_k = \{(z, w) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid w^{k+1} = z^k(z^2 - 1)\}$$

and

$$M_k = \overline{M} \setminus \{p_{-1}, p_1, p_\infty\},$$

where

$$p_{-1} = (-1, 0),\ \ p_1 = (1, 0),\ \ p_\infty = (\infty, \infty)$$

and

$$\omega = \left(\frac{z}{w}\right)^k dz = \frac{w}{z^2 - 1} dz, \qquad g = c/w.$$

Example: Gama - ——, immersed.

Example (5)

E. S. Gama - ——:

There are complete minimal surfaces $T_k^2 \setminus X$, $\sharp X = 4$, with finite total curvature and Gauss map missing a par of antipode pints for all genus k.

Consider

$$\overline{M}_k = \{(z,w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}; w^{k+1} = z^k(z^2-1)\}$$

and

$$M_k = \overline{M}_k \setminus \{p_1, p_2, p_3, p_4\},$$

where

$$p_1 = (0,0),\ p_2 = (1,0),\ p_3 = (-1,0),\ p_4 = (\infty,\infty)$$

Three family of examples.

Theorem

1. $g = c\frac{1}{z^j w}$, $\omega = \frac{z^k}{w^k} dz$, *for* $j \geq 1$ *odd,*
2. $g = c\frac{1}{z^j w}$, $\omega = \frac{z^{j+k}}{w^k} dz$, *for* $j \geq 2$ *even*

Then there exist unique c such that the parameters above are the Weierstrass-Enneper parameters of a complete minimal immersion of total curvature $4\pi((k+1)j + k + 2)$ *with four ends and genus k such that the Gauss map misses two antipodes directions.*

Theorem

$$g = c\frac{z^{2j+1}}{w}, \quad \omega = \left(\frac{z}{w}\right)^k \frac{dz}{z^{2j}},$$

where $j \in \mathbb{N}$*. Then there exist unique* $c \in \mathbb{R}$ *such that the parameters above is the Weierstrass-Enneper parameters of a complete minimal immersion of total curvature* $4\pi[(2j+1) + 2jk]$ *with four ends and genus k whose Gauss map misses two antipodal points.*

1960'th Robert Osserman, basic tools.

Let M a complete minimal surface immersed into $\mathbb{R}^3$ with Gauss map G and missing set of points in the image $Y = \mathbb{S}^2 \setminus G(M)$.

(a) Y has capacity zero.

If the total curvature of M is finite:

(b) M is conformal to a surface of $\overline{M} \setminus X$ where $\overline{M}$ is compact and $X \subset \overline{M}$ is finite.

(c) The Guass map extends to a regular map $G\colon \overline{M} \to \mathbb{S}^2$

(d) $\sharp Y \leq 3$

Example

The catenoid has $\sharp Y = 2$.
There are not know examples with $\sharp Y = 3$, complete and finite total curvature.

Selected results: Xavier and Fujimore.

Xavier theorem:

$$\sharp Y \leq 6.$$

Fujimore result:

$$\sharp Y \leq 4$$

About finite total curvature: Osserman

$$\sharp Y \leq 3$$

Curiosity: (interesting theorem)
A. Alarcón, F. Forstnerič, and F. J. López

Every meromorphic function in a open Riemann surface
is the Gauss map of a conformal minimal surface

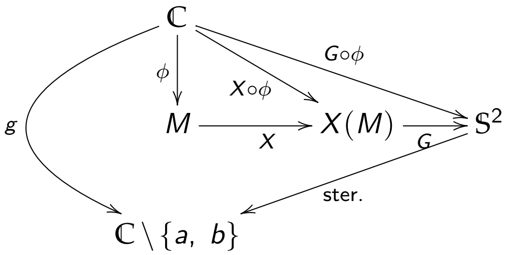
Recent results.

Theorem (Jorge - Mercuri,)

The Gauss map of a complete non-flat minimal surfaces in $\mathbb{R}^3$ *with finite total curvature can omit at most 2 points of the sphere.*

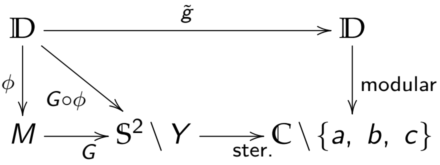
Proof of th. Jorge - Mercuri.

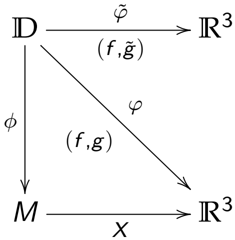
The universal covering of M is conformally equivalent to $\mathbb{C}$ or $\mathbb{D}$.
If $\sharp Y = 3$ the universal covering is not $\mathbb{C}$.



Proof of th. Jorge - Mercuri.

If $\sharp Y = 3$ the universal covering is $\mathbb{D}$.





Proof of th. Jorge - Mercuri.

$$\text{Poles of } g = \text{Poles of } \tilde{g}$$

with the same order of branching.
Then the par $(f, \tilde{g})$ is a Weierstrass par of some minimal immersion $\tilde{\varphi}$.
The assumption that E is some compact subset of $\overline{M}_\mu$, with finite number of connected components is equivalently to say that the total curvature is finite or the degree of G is finite.

Lemma

Let M be a complete non flat minimal surface whose Gauss map extends continuously to a map $G\colon \overline{M}_\mu \to \mathbb{S}^2$, $M = \overline{M}_\mu \setminus E$, and finite degree. If the set of missing points Y has $\sharp Y = 3$, or 4 then $\tilde{\varphi}\colon \mathbb{D} \to \mathbb{R}^3$ is a complete minimal immersion with image of the Gauss map inside a half sphere. In particular such immersion does not exist.

Proof of th. Jorge - Mercuri.

From now on $\sharp Y = 3$ or 4. Set $Y = \{y_1,\ y_2,\ y_3,\ y_4\}$ for $\sharp Y = 4$ and for $\sharp Y = 3$ the point y_4 do not exist. Consider the minimal immersion $X\colon M \to \mathbb{R}^3$ in position with $y_0 = (0,0,1) \notin Y$. We set

1. $B_{\delta_0} = \{z \in M \mid \mathrm{dist}_{\delta}^{\mathbb{S}^2}(y_0,\ G(z)) < \delta_0\}$ and $B_\delta \cap G^{-1}(Y) = \emptyset$,
2. There is a constant $c_1 > 0$ such that $\mathrm{dist}^M\big(\partial U_\delta,\ \partial U_{\delta'}\big) \geq c_1 > 0$ for all $0 < \delta' < \delta \geq \delta_0$.

The item (2) is true since $B_\delta \setminus B_{\delta'}$ is precompact and has finite connected components.

Proof of th. Jorge - Mercuri.

Let $\gamma(t)$, $0 \leq t < \omega_0 \leq \infty$, be a divergent curve in $\mathbb{D}$ parametrized by the arc length of the plane. The length $\tilde{\ell}(\gamma)$ of the immersion $\tilde{\varphi}$ satisfies

$$\tilde{\ell}(\gamma) = \int_0^{\omega_0} |f|(1 + |\tilde{g}|^2)|d\gamma| \geq \int_0^{\omega_0} |f||d\gamma|.$$

Then if $\int_0^{\omega_0} |f||d\gamma| = \infty$ we get $\tilde{\ell}(\gamma) = \infty$. Here ω_0 is the length of γ with the Euclidean metric.

Proof of th. Jorge - Mercuri.

Then we have $\tilde{\ell}(\gamma) = \infty$ in the following situations.

(i) *There is* $\delta,\ 0 < \delta \leq \delta_0$, *such that* $\phi(\gamma(t)) \in M \setminus B_\delta$ *for all* $t \geq t_0 \geq 0$..

(ii) *There are increasing sequences* $t_j < s_j < t_{j+1} \cdots$ *such that* $\phi(\gamma(t_j)) \in B_{\delta'}$ *and* $\phi(\gamma(s_j)) \notin B_\delta$ *for* $0 < \delta' < \delta \leq \delta_0$.

Proof of th. Jorge - Mercuri.

In item (i) we have g bounded over the image of $\alpha = \phi(\gamma)$ implying

$$c_\delta = \sup_{z \in M \setminus B_\delta} |g(z)| < \infty.$$

Then

$$\int_\gamma |f||dz| \geq \frac{1}{1 + c_\delta^2} \int_{t_0}^{\omega_0} |f|(1 + |g|^2)|d\gamma| = \infty, \qquad (1)$$

and we get $\tilde{\ell}(\gamma) = \infty$.

Proof of th. Jorge - Mercuri.

For item (ii) the curve goes inside $B_{\delta'}$ and outside B_δ infinite number of times. By item (2) we get

$$\int_{t_k}^{s_k} |f||d\gamma| \geq \frac{1}{1+c_\delta^2}\int_{t_k}^{s_k} |f|(1+|g|^2)|d\gamma| \geq \frac{c_1}{1+c_\delta^2}.$$

Then

$$\int_{t_k}^{s_{m+k}} |f||d\gamma| \geq \frac{1}{1+c_\delta^2}\sum_{j=0}^{j=m}\int_{t_{k+j}}^{s_{k+j}} |f|(1+|g|^2)|d\gamma| \geq \frac{c_1}{1+c_\delta^2}m, \quad (2)$$

implying $\tilde{\ell}(\gamma) = \infty$.

Proof of th. Jorge - Mercuri.

Observe that $\phi^{-1}(y_0)$ is a sequence of points into $\mathbb{D}$ accumulating only at $\partial\mathbb{D}$. If γ is divergent in $\mathbb{D}$ we can not have $\lim_t \phi(\gamma(t)) = y_0$ otherwise $\gamma(t)$ is in one connected component of $\phi^{-1}(B_\delta)$ for all $t \geq t_0 > 0$ and is not divergent. The unique way to have y_0 accumulation point of $\phi(\gamma)$ is if item (ii) happen. We already know that in this case $\tilde{\ell}(\gamma) = \infty$. This finish the proof of the completeness.

Proof of th. Jorge - Mercuri.

Barbosa-do Carmo:

If the area of $G(M) \subset \mathbb{S}^2$ is at most 2π then M is globally stable.

do Carmo - Peng, Shoen:

Minimal surfaces complete and globally stable are flat.

Xavier, Fujimore:

$$\sharp\Big(\mathbb{S}^2 \setminus G(M)\Big) > 4 \text{ implies } M \text{ flat.}$$

Osserman:

$$\mathbb{S}^2 \setminus G(M) \text{ has zero capacity.}$$

Proof of th. Jorge - Mercuri.

Remark

A complete minimal surfaces satisfying the conditions of lemma (11) does not exist. In fact we have proved that for all divergent curve γ into $\mathbb{D}$ we have $\int_\gamma |f||dz| = \infty$. By a well know Osserman's lemma there is no holomorphic f with finite number of zeros and $|f||dz|$ complete a metric in the unit disc.
Our f has infinite number of zeros and do not exist for other reason.

Generalization of little Picard theorem

Theorem (de Andrade - ——:)

Let $M = \overline{M} \setminus E_m$ be a finite geometric type surface and $F : M \to \mathbb{S}^2$ be a non constant branched covering with finite fiber that has a C^0 extension to a branched covering $F : \overline{M} \to \mathbb{S}^2$. Then $\sharp\left(\mathbb{S}^2 \setminus F(M)\right) \leq 2$. In particular if G is the Gauss map of M and M not flat then $\sharp\left(\mathbb{S}^2 \setminus G(M)\right) \leq 2$.

Corollary (Generalization of little Picard theorem)

If $\overline{M} = \mathbb{S}^2$ then $\sharp(\mathbb{S}^2 \setminus F(M)) \leq 2$.

Finite Geometric Type

Finite geometric type surfaces was introduced in Barbosa - Fukuoka - Mercuri's paper as those immersions $\varphi\colon M \to \mathbb{R}^3$ of a surface M such that M is complete in the induced metric and

1. M is diffeomorphic to a compact oriented surface $\overline{M}$ minus a finite set of points, $E_m = \{w_1, \dots, w_m\}$,
2. the Gaussian curvature vanishes only at a finite number of points,
3. the Gauss map G extends to a smooth branched covering, denoted by the same symbol, $G\colon \overline{M} \to \mathbb{S}^2$.

Proof of th. de Andrade - Jorge

Lemma

Given three points $a_1 = \infty,\ a_2 = 0,\ x_1,\ x_1 \notin \{a_1,\ a_2\}$ *of* $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$, *the points* $x_2 = -3x_1,\ y_1 = -x_1,\ y_2 = 3x_1,\ w = 16x_1^3$ *and the rational function*

$$f(z) = \frac{(z - x_1)^3 (z - x_2)}{z}, \quad z \in \mathbb{S}^2 \tag{3}$$

then

$$\deg f = 4, \quad \beta_f(a_j) = \beta_f(x_j) = \beta_f(y_j) = \begin{cases} 2, & j = 1, \\ 0, & j = 2 \end{cases}$$

In particular if $X = \{0,\ \infty,\ x_1,\ x_2,\ y_1,\ y_2\}$ *then*

$$f \colon \mathbb{S}^2 \setminus X \to \mathbb{S}^2 \setminus \{0,\ w,\ \infty\} \tag{4}$$

is a regular covering map of degree 4 *and* $\sharp X = 6$.
The map $f_\star \colon \pi_1(\mathbb{S}^2 \setminus X) \to \pi_1(\mathbb{S}^2 \setminus Y)$ *is over.*

Proof of th. de Andrade - Jorge

Reciprocally, given a branched covering $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ and subsets Y, $X = h^{-1}(Y)$, such that

1. The set B_h of branching points of h is a subset of X,
2. $h_\star\Big(\pi_1(\mathbb{S}^2 \setminus X)\Big) = \pi_1\Big(\mathbb{S}^2 \setminus Y\Big)$

then $\sharp Y = 3 + m$, $m \geq 0$ an integer, $\sharp X = 6 + 4m$, $\beta_h(x) = 2$ for all $x \in B_h$, $\deg h = 4$, and $\sharp B_h = 3$. We have splits $X = X_0 \cup X_1$, $Y = Y_0 \cup Y_1$ with $\sharp X_0 = 6$, $\sharp Y_0 = 3$, $\sharp X_1 = \sharp Y_1 = m$, and $B_h \subset X_0$.

Further, if ψ is the Möbius transform such that $\psi(Y_0) = \{0,\ w,\ \infty\}$, and $X_f = f^{-1}(\psi(Y))$, where f is defined by the choose $x_1 = (w/16)^{1/3}$, then there is a diffeomorphism preserving fiber $\phi\colon \mathbb{S}^2 \setminus X_f \to \mathbb{S}^2 \setminus X$ such that $\psi \circ h \circ \phi = f$.

Proof of th. de Andrade - Jorge

Let D_1, D_2, and D_3 be 3 disks without the center and $f\colon \overline{D}_1 \to \overline{D}_2$ and $F\colon \overline{D}_3 \to \overline{D}_2$ branched covering maps with one branching at the center of each disk of order β_f and β_F. We consider f and F of class C^l, $l \geq 2$, inside D_j and continuous in $\overline{D}_j$. The first homotopy group $\pi_1(D_j)$ is an infinite cyclic group with generators γ_j. The existence of a continuous lift $\tilde{F}\colon D_3 \to D_1$ of F by f is equivalent to

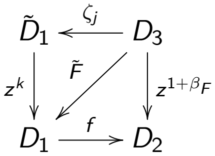
$$F_\star\Big(\pi_1(D_3)\Big) \subset f_\star\Big(\pi_1(D_1)\Big),$$

where the subindex means the induced group homomorphism between the fundamental groups. Since $f_\star[\gamma_1] = [\gamma_2]^{1+\beta_f}$ and $F_\star[\gamma_3] = [\gamma_2]^{1+\beta_F}$ the existence of $\tilde{F}$ is equivalent to have

$$f_\star[\gamma_1]^k = F_\star[\gamma_3], \quad k \in \mathbb{Z}, \quad k \geq 1,$$

or yet $1 + \beta_F = k(1 + \beta_f)$, $k \geq 1$.

Proof of th. de Andrade - Jorge



Lemma

Let D_1, D_2, and D_3 be 3 disks without the center and $f\colon \overline{D}_1 \to \overline{D}_2$ and $F\colon \overline{D}_3 \to \overline{D}_2$ branched covering maps with one branching at the center of each disk of order β_f and β_F. We consider f and F of class C^l, $l \geq 2$, inside D_j and continuous in $\overline{D}_j$. Then there exist a continuous lifting $\tilde{F}\colon \overline{D}_3 \to \overline{D}_1$ iff

$$\frac{1+\beta_F}{1+\beta_f} = k = 1 + \beta_{\tilde{F}}, \quad k \in \mathbb{Z}, \quad k \geq 1.$$

In that case there exist a k-root $\zeta_j\colon D_3 \to \tilde{D}_1$ of $\tilde{F}$ such that $\zeta_j^k = \tilde{F}$ and $\tilde{F}$ has the same differentiability of f and F.

Proof of th. de Andrade - Jorge

Theorem

*Let $M = \overline{M} \setminus E_m$ be a finite geometric type surface. Let $F\colon M \to \mathbb{S}^2$ be an at least C^2 branched covering map having C^0 extension to a branched covering map denoted by $F\colon \overline{M} \to \mathbb{S}^2$. Then the (**T-C**) and (**R-H**) formulas holds*

$$2\deg F = -\chi(\overline{M}) + \sharp E_m + I(E_m), \quad (5)$$

$$2\deg F = \chi(\overline{M}) + \beta_F(\overline{M}). \quad (6)$$

In particular

$$\sharp Y \leq 3$$

for $Y = \mathbb{S}^2 \setminus F(M)$.

Proof of th. de Andrade - Jorge

To get the Riemann-Hurwirtz formula we endowed M with the metric of $\mathbb{S}^2$ by F and calculate the total curvature.

Proof of th. de Andrade - Jorge

The proof of th. de Andrade-Jorge

Let $M = \overline{M} \setminus E_m$ be a finite geometric type surface and $G\colon M \to \mathbb{S}^2$ be a non constant branched covering with finite fiber that has a C^0 extension to a branched covering $G\colon \overline{M} \to \mathbb{S}^2$. Set $Y = \left(\mathbb{S}^2 \setminus G(M)\right)$ and suppose that $\sharp Y = 3$. Hence it is possible to find $h\colon \mathbb{S}^2 \setminus X \to \mathbb{S}^{\setminus} Y$ satisfying the lemma and proving the existence of the lifting $F\colon M \to \mathbb{S}^2 \setminus X$ of G by h and F has continuous extension to $\overline{M}$. Since $X = \mathbb{S}^2 \setminus F(M)$ and $\sharp X = 6$ we get a contradiction with theorem.

Applications (de Andrade - Jorge)

Theorem (A)

Let M be one of the following:

(1) *a complete Bryant surface with finite total curvature or,*

(2) *a complete algebraic Bryant surface with the dual metric or,*

(3) *a complete CMC-1 face of finite type with elliptic ends and dual metric.*

Then the hyperbolic Gauss map G of M omit at most 2 points, otherwise G is constant. Further, in case of constant hyperbolic Gauss map M is a horosphere for items (1) and (2) and a horosphere space like type surface for item (3).

All results in this theorem are sharp.

Applications (de Andrade - Jorge)

Theorem (B)

Let M be one complete surface of type:

(1) *minimal surface with finite total curvature into $\mathbb{R}^3$;*

(2) *Algebraic Bryant surface with finite total curvature;*

(3) *Algebraic Bryant surface endowed with the dual metric;*

(4) *CMC-1 Face surface of finite type, elliptic ends endowed with the lift metric.*

Then M is parabolic. If M has a Kähler structure then all bounded holomorphic map $g\colon M \to \mathbb{C}$ is constant.

Applications

All surfaces in items (1), (2), (3), (4), are isometric to a complete minimal surface M into $\mathbb{R}^3$ with finite total curvature. Then they have a natural Kähler structure. If the Gauss map G of M miss at least one point $y \in \mathbb{S}^2$ we can rotated M inside $\mathbb{R}^3$ to making $y = (0,\ 0,\ 1)$. If $g\colon M \to \mathbb{C}$ is the stereographic projection of G then g is holomorphic. If the Gauss map miss one more point we can not do a lift to a bounded holomorphic map unless M is simply connected. All surfaces described in the theorem B are Kähler.

Applications

All surfaces in items (1), (2), (3), (4), are isometric to a complete minimal surface M into $\mathbb{R}^3$ with finite total curvature. Then they have a natural Kähler structure. If the Gauss map G of M miss at least one point $y \in \mathbb{S}^2$ we can rotated M inside $\mathbb{R}^3$ to making $y = (0,\ 0,\ 1)$. If $g\colon M \to \mathbb{C}$ is the stereographic projection of G then g is holomorphic. If the Gauss map miss one more point we can not do a lift to a bounded holomorphic map unless M is simply connected. All surfaces described in the theorem B are Kähler.

The decay of radial curvature is of kind $K(p) \geq -\rho(p)^{-(2+\epsilon)}$ implying volume growth of kind ρ^2 and then parabolic.

THE END

Thanks for the attention!